Cauchy's n^{th} root Test.

— Let Σu_n be a positive term Series and let $\lim_{n\to\infty} [u_n]^{1/n} = l$

Then, the series is

i) Convergent if $l<1$
2) Divergent if $l>1$
3) No firm decision is possible if $l=1$

Proof:

Case I:

Suppose $l<1$

Let ρ be any number such that
$$l<\rho<1$$
then $\exists$ m such that $\forall$ $n \geq m$
$$[u_n]^{1/n} < \rho$$
$$\Rightarrow \quad u_n < \rho^n$$
Now, ρ being less than 1 the geometric series $\Sigma \rho^n$ is convergent.
Thus, by Comparision Test of the first Type, it follows that Σu_n is convergent.

Case II:

Suppose $l > 1$

Let α be any number such that
$1 < \alpha < l$

then $\exists$ $m \in N$ such that $\forall$ $n \geq m$

$$[u_n]^{1/n} > \alpha$$

$$\Rightarrow \quad u_n > 1 \qquad \forall \; n \geq m.$$

It follows that given series is divergent.

Case III:

Suppose $l = 1$

Consider the two series

i) $\sum \frac{1}{n}$

ii) $\sum \frac{1}{n^2}$.

We know that series (i) is divergent and series (ii) is Convergent.

but $\lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/n} = 1$

Also, $\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = 1$

Thus, here, we have Proved examples of two series for each of which $\lim_{n\to\infty} [u_n]^{1/n} = 1$

but while one series is convergent and the other is divergent

Q). Test the convergence of the series

i) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Here, $u_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$\therefore \lim_{n\to\infty} [u_n]^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$= \frac{1}{e} = \frac{1}{2.7} < 1$$

So, by Cauchy's n^{th} root test the series is convergent.

ii) $\sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}$

Here,

$$u_n = \frac{n^{n^2}}{(n+1)^{n^2}} = \left\{\frac{n^n}{(n+1)^n}\right\}^n$$

$$\therefore \lim_{n\to\infty} [u_n]^{1/n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$$= \frac{1}{e} < 1$$

So, by Cauchy's nth root test the series is convergent

iii) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$

Here,

$$u_n = \left\{\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right\}^{-n}$$

$$= \left\{\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right\}^{-n}$$

$$\therefore \lim_{n\to\infty} [u_n]^{1/n} = \lim_{n\to\infty} \left\{\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right\}^{-1}$$

$$= \lim_{n\to\infty} \left\{\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)^{n} - \frac{n+1}{n}\right\}^{-1}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-1} \left\{\left(\frac{n+1}{n}\right)^{n} - 1\right\}^{-1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-1} \left\{\left(1+\frac{1}{n}\right)^{n} - 1\right\}^{-1}$$

$$= 1\,(e-1)^{-1}$$

$$= \frac{1}{e-1}$$

$$< 1$$

So, by Cauchy's nth root test the series is convergent.